Int. J. Solids Structures, 1977, Vol. 13, pp. 105–123. Pergamon Press. Printed in Great Britain

TOWARDS THE SOLUTION OF FINITE PLANE-STRAIN PROBLEMS FOR COMPRESSIBLE ELASTIC SOLIDS

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(*Received* 18 *February* 1976; *revised* 6 *July* 1976)

Abstract—A new approach to the solution of finite plane-strain problems for *compressible* isotropic elastic solids is considered. The general problem is formulated in terms of a pair of deformation invariants different from those normally used, enabling the components of (nominal) stress to be expressed in terms of four functions, two of which are rotations associated with the deformation. Moreover, the inverse constitutive law can be written in a simple form involving the same two rotations, and this allows the problem to be formulated in a dual fashion.

For particular choices of strain-energy function of the elastic material solutions are found in which the governing differential equations partially decouple, and the theory is then illustrated by simple examples. It is also shown how this part of the analysis is related to the work of F. John on harmonic materials.

Detailed consideration is given to the problem of a circular cylindrical annulus whose inner surface is fixed and whose outer surface is subjected to a circular shear stress. We note, in particular, that material circles concentric with the annulus and near its surface *decrease* in radius whatever the form of constitutive law within the given class. Whether the volume of the material constituting the annulus increases or decreases depends on the form of law and the magnitude of the applied shear stress.

1. INTRODUCTION

This paper describes an attempt to obtain direct solutions of boundary-value problems in plane-strain finite elasticity, that is direct in the sense that no assumption is made *ab initio* about the form of the deformation or stress (except that attention is restricted to plane-strain). This distinguishes the present approach from that adopted in the inverse or semi-inverse methods described in, for example, Truesdell and Noll[1] and Green and Adkins[2]. Here, only isotropic elastic materials are considered and, initially, no restriction is put on the form of the constitutive law employed. More specifically, we deal with *compressible* materials, our intention in part being to analyse the effect of allowing volume changes.

The majority of work in finite elasticity adopts the simplifying assumption of incompressibility, and very few non-trivial analytical solutions to boundary-value problems for compressible materials have been found. Of course, *numerical* solutions are obtainable (in principle, for any form of constitutive law) as evidenced by the book of Oden[3]. This work is very valuable, but its objectives are somewhat different from ours. We attempt to gain insight into some of the fundamental problems in finite elasticity by introducing certain analytical techniques appropriate to the discussion of compressible materials (similar techniques can be adopted for incompressible materials), going as far as possible towards obtaining analytical solutions of specific boundary-value problems. We use numerical results ultimately, but for purposes of illustration only. The two approaches are complementary.

Let $\mathbf{X}$ and $\mathbf{x}$ respectively denote the position vectors of a typical material point in the undeformed and deformed configurations of the material, the undeformed configuration being assumed stress free. Rectangular Cartesian coordinates are used throughout and the components of $\mathbf{X}$ and $\mathbf{x}$ are denoted by X_μ $(\mu = 1, 2, 3)$ and x_i $(i = 1, 2, 3)$.

Since attention is to be restricted to plane strain we write $x_3 = X_3$, it being convenient to assume that there is no uniform strain in the third direction (although there is no difficulty in incorporating such a strain if required). Also x_1 and x_2 depend only on X_1 and X_2.

The plane part of the deformation gradient $\partial x_i/\partial X_\mu$ $(i, \mu = 1$ or $2)$ is denoted by

$$\boldsymbol{\alpha} \equiv \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$$

in symbolic notation.

The invariants of $\boldsymbol{\alpha}$ given by

$$\left.\begin{aligned} I &= \tfrac{1}{2}(\alpha_{11}^2 + \alpha_{12}^2 + \alpha_{21}^2 + \alpha_{22}^2), \\ J &= \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} \end{aligned}\right\} \tag{1}$$

are those commonly employed in problems of isotropic elasticity theory, being essentially the principal invariants of the Green deformation measure. In the present paper, however, it will be found advantageous to use different invariants, denoted p and q and defined by

$$\left.\begin{aligned} p &= \{2(I+J)\}^{1/2} = \{(\alpha_{11}+\alpha_{22})^2 + (\alpha_{12}-\alpha_{21})^2\}^{1/2}, \\ q &= \{2(I-J)\}^{1/2} = \{(\alpha_{11}-\alpha_{22})^2 + (\alpha_{12}+\alpha_{21})^2\}^{1/2}, \end{aligned}\right\} \tag{2}$$

where $p > 0$ and $q \geq 0$.

Let $\boldsymbol{\beta}$ be the inverse of $\boldsymbol{\alpha}^T$, where $\boldsymbol{\alpha}^T$ is the transpose of $\boldsymbol{\alpha}$. We note, in particular, that

$$\frac{\partial J}{\partial \boldsymbol{\alpha}} \equiv J\boldsymbol{\beta}^T \equiv \begin{pmatrix} \alpha_{22} & -\alpha_{12} \\ -\alpha_{21} & \alpha_{11} \end{pmatrix}. \tag{3}$$

In Section 2 the elastic stress-deformation relation is expressed in terms of the invariants p and q and two rotation angles. The components of (nominal) stress are then written in terms of four functions, denoted A, B, C and D, which are related through four first-order partial differential equations, namely the equilibrium equations and compatibility conditions. This approach requires that the stress-deformation relation be invertible in a sense to be described. The dual problem, in which the components of the deformation gradient are expressed in terms of four functions, is formulated in an analogous manner in Section 3.

Choices of the functions A, B, C and D may be made so that the governing equations partially decouple, the resulting equations necessarily putting restrictions on the class of constitutive laws that can then be admitted. For example, A, B, C and D may be chosen to be harmonic functions. The resulting constitutive laws include those of the class of so-called harmonic materials discussed by John [4]. The method used by John differs from that presented here, being in a sense less general while for harmonic materials it is more general. These materials are discussed in Sections 4 and 5. Inevitably, in making assumptions about A, B, C and D questions of existence of solutions to particular boundary-value problems are raised, and this is taken into account.

In Section 6 the solution obtained by the present approach is given for a simple mixed boundary-value problem and compared with the corresponding solution in the linear theory. For a specific form of constitutive law detailed calculations have been performed and these are discussed in Section 7.

2. CONSTITUTIVE RELATIONS AND GOVERNING EQUATIONS

We consider elastic solids which possess a stored-energy function W per unit volume in the undeformed configuration. Since attention is being restricted to plane strain W can be regarded as a function of $\boldsymbol{\alpha}$, and $W(\boldsymbol{\alpha})$ is the strain energy per unit area in the undeformed configuration of the (1, 2)-plane, and per unit length in the X_3-direction.

Let

$$\mathbf{s} \equiv \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$

denote the in-plane components of the nominal stress (the transpose of the first Piola–Kirchhoff stress). Then

$$\mathbf{s} = \frac{\partial W}{\partial \boldsymbol{\alpha}}, \; s_{\mu i} = \frac{\partial W}{\partial \alpha_{i\mu}} \, (i, \mu = 1 \text{ or } 2), \tag{4}$$

and the only other non-zero component of stress is s_{33}, required to maintain the plane-strain condition $x_3 = X_3$.

In plane strain the equilibrium equations are

$$\left.\begin{aligned} s_{11,1} + s_{21,2} = 0, \\ s_{12,1} + s_{22,2} = 0, \end{aligned}\right\} \tag{5}$$

when there are no body forces, and the compatibility equations are

$$\left.\begin{aligned} \alpha_{11,2} - \alpha_{12,1} = 0, \\ \alpha_{21,2} - \alpha_{22,1} = 0, \end{aligned}\right\} \tag{6}$$

where $_{,\mu}$ denotes $\partial/\partial X_\mu$ $(\mu = 1, 2)$.

For the isotropic elastic solids (that is isotropic relative to the stress free configuration) to which this paper is devoted W can be expressed as a function only of two independent invariants of $\boldsymbol{\alpha}$ (provided the material is homogeneous). For example, when W is regarded as a function of the invariants I and J given by (1), (4) becomes

$$\mathbf{s} = W_I \boldsymbol{\alpha}^T + J W_J \boldsymbol{\beta}^T, \tag{7}$$

where $W_I = \partial W/\partial I$, $W_J = \partial W/\partial J$.

Incidentally, with the help of (3), the universal relations

$$\frac{\alpha_{12} - \alpha_{21}}{\alpha_{11} + \alpha_{22}} = \frac{s_{21} - s_{12}}{s_{11} + s_{22}}, \frac{\alpha_{11} - \alpha_{22}}{\alpha_{12} + \alpha_{21}} = \frac{s_{11} - s_{22}}{s_{12} + s_{21}} \tag{8}$$

are easily established from (7) provided

$$W_I \neq \pm W_J. \tag{9}$$

From (2) we obtain

$$\left.\begin{aligned} \frac{\partial p}{\partial \alpha_{11}} = \frac{\partial p}{\partial \alpha_{22}} = \frac{\alpha_{11} + \alpha_{22}}{p}, \frac{\partial p}{\partial \alpha_{12}} = -\frac{\partial p}{\partial \alpha_{21}} = \frac{\alpha_{12} - \alpha_{21}}{p}, \\ \frac{\partial q}{\partial \alpha_{11}} = -\frac{\partial q}{\partial \alpha_{22}} = \frac{\alpha_{11} - \alpha_{22}}{q}, \frac{\partial q}{\partial \alpha_{12}} = \frac{\partial q}{\partial \alpha_{21}} = \frac{\alpha_{12} + \alpha_{21}}{q}, \end{aligned}\right\} \tag{10}$$

and we note that (9) is equivalent to

$$W_p \neq 0, \; W_q \neq 0. \tag{11}$$

When W is expressed as a function of the invariants p and q (4) can be written as

$$\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = W_p \begin{pmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\chi \end{pmatrix} + W_q \begin{pmatrix} \cos\chi^* & \sin\chi^* \\ \sin\chi^* & -\cos\chi^* \end{pmatrix}, \tag{12}$$

where

$$\left.\begin{aligned} \cos\chi = \frac{\alpha_{11} + \alpha_{22}}{p}, \sin\chi = \frac{\alpha_{12} - \alpha_{21}}{p}, \\ \cos\chi^* = \frac{\alpha_{11} - \alpha_{22}}{q}, \sin\chi^* = \frac{\alpha_{12} + \alpha_{21}}{q}. \end{aligned}\right\} \tag{13}$$

The rotation

$$\begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix}$$

is in fact the local rotational part of the deformation, as may be demonstrated by showing, with the help of (13), that

$$\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}\begin{pmatrix}\cos\chi & -\sin\chi\\ \sin\chi & \cos\chi\end{pmatrix} \tag{14}$$

is symmetric (by the polar decomposition theorem).

The angle χ^* does not have such an immediate physical interpretation, but it is related to χ by

$$\chi^* - \chi = 2\theta_E, \tag{15}$$

where $\theta_E(0 \le \theta_E \le \frac{1}{2}\pi)$ is the orientation of the principal axes of the Eulerian strain ellipsoid, that is the principal axes of (14). In the case of circular symmetry, for example, $\chi = 0$, $\theta_E = \theta$ and $\chi^* = 2\theta$, where θ is the polar angle. And in simple shear $\chi^* = \frac{1}{2}\pi$.

At this point it is important to note that the stress–strain law of an elastic material having a strain-energy function can also be written as

$$\boldsymbol{\tau} = \frac{\partial W}{\partial \mathbf{u}}, \tag{16}$$

where $\mathbf{u}$ is the (positive definite and symmetric) right stretch matrix such that $\boldsymbol{\alpha}^T \times \boldsymbol{\alpha} = \mathbf{u}^2$, recalling the definition of $\boldsymbol{\alpha}$ in plane strain. The symbol $\times$ denotes the usual matrix product. If we use the polar decomposition theorem to write $\boldsymbol{\alpha} = \mathbf{r} \times \mathbf{u}$, where $\mathbf{r}$ is proper orthogonal, then, for isotropic elastic solids, $\boldsymbol{\tau} = \mathbf{s} \times \mathbf{r}$. For our purposes (16) may be taken as the definition of the (symmetric) Biot stress $\boldsymbol{\tau}$, the conjugate of $\mathbf{u}$. This particular stress measure plays a central role in considerations relating to the invertibility of the stress–strain law (4), as has been discussed in some detail by Ogden[5].

Let λ_1 and λ_2 denote the principal stretches, that is the principal components of $\mathbf{u}$. Then, if τ_1 and τ_2 are the principal components of $\boldsymbol{\tau}$ in plane-strain, the principal components of (16) are

$$\tau_1 = \frac{\partial W}{\partial \lambda_1}, \tau_2 = \frac{\partial W}{\partial \lambda_2}. \tag{17}$$

Hence with the help of $I = \frac{1}{2}(\lambda_1{}^2 + \lambda_2{}^2)$ and $J = \lambda_1\lambda_2$, we obtain

$$\tau_1 + \tau_2 = (W_I + W_J)(\lambda_1 + \lambda_2), \tau_1 - \tau_2 = (W_I - W_J)(\lambda_1 - \lambda_2).$$

This shows that conditions (9) or (11) can be expressed as

$$\tau_1 + \tau_2 \neq 0 \quad \text{or} \quad \tau_1 - \tau_2 \neq 0 \quad (\lambda_1 \neq \lambda_2). \tag{18}$$

The restrictions (18) are relevant to the discussion of the invertibility of the stress-deformation relation in the form (4) or (16), [5].

Expressed in terms of the principal stretches the invariants p and q are

$$p = \lambda_1 + \lambda_2, \quad q = |\lambda_1 - \lambda_2|. \tag{19}$$

We now define stress invariants, denoted by $\bar{p}$ and $\bar{q}$, by

$$\left.\begin{aligned}\bar{p} &= \{(s_{11} + s_{22})^2 + (s_{12} - s_{21})^2\}^{1/2},\\ \bar{q} &= \{(s_{11} - s_{22})^2 + (s_{12} + s_{21})^2\}^{1/2}.\end{aligned}\right\} \tag{20}$$

In terms of the principal components of $\boldsymbol{\tau}$ these are

$$\bar{p} = (\tau_1 + \tau_2), \quad \bar{q} = |\tau_1 - \tau_2|, \tag{21}$$

but, in contrast to p, *$\bar{p}$ may be positive or negative.*

It is easily seen from (8), (13) and (20) that we can write

$$\left.\begin{aligned}\cos\chi &= \pm\frac{s_{11}+s_{22}}{\bar{p}}, & \sin\chi &= \pm\frac{s_{21}-s_{12}}{\bar{p}},\\ \cos\chi^* &= \pm\frac{s_{11}-s_{22}}{\bar{q}}, & \sin\chi^* &= \pm\frac{s_{12}+s_{21}}{\bar{q}},\end{aligned}\right\} \tag{22}$$

the signs being chosen so that (13) and (22) are compatible.

We now introduce the notation

$$\left.\begin{aligned}A &= W_p\cos\chi, & B &= W_p\sin\chi,\\ C &= W_q\cos\chi^*, & D &= W_q\sin\chi^*,\end{aligned}\right\} \tag{23}$$

so that eqns (12) may be written in the compact form

$$\left.\begin{aligned}s_{11} &= A+C, & s_{12} &= -B+D,\\ s_{21} &= B+D, & s_{22} &= A-C.\end{aligned}\right\} \tag{24}$$

It follows from (20), (23) and (24) that

$$\tfrac{1}{2}\bar{p} = W_p = (A^2+B^2)^{1/2}, \quad \tfrac{1}{2}\bar{q} = W_q = (C^2+D^2)^{1/2}. \tag{25}$$

The connections

$$\tan\chi = \frac{\alpha_{12}-\alpha_{21}}{\alpha_{11}+\alpha_{22}} = \frac{s_{21}-s_{12}}{s_{11}+s_{22}} = \frac{B}{A}, \; W_p \neq 0, \tag{26}$$

$$\tan\chi^* = \frac{\alpha_{12}+\alpha_{21}}{\alpha_{11}-\alpha_{22}} = \frac{s_{12}+s_{21}}{s_{11}-s_{22}} = \frac{D}{C}, \; W_q\neq 0 \quad \text{for} \quad q\neq 0, \tag{27}$$

are now noted for future reference.

If s is given, the angles χ and χ^* are determined from (26) and (27) only to within an integral multiple of π except in the cases $\bar{p}=0$, when χ is indeterminate, and $\bar{q}=0$ for $q\neq 0$, when χ^* is indeterminate. In other words, with the exception of the cases mentioned above, there are two possible polar decompositions of the form $\mathbf{s} = \boldsymbol{\tau}\times\mathbf{r}^T$ for a given s in plane strain, r being proper orthogonal and $\boldsymbol{\tau}$ symmetric. Hence, *in plane strain*, a given s corresponds to (at least) two distinct $\boldsymbol{\alpha}$'s.

Since r is proper orthogonal

$$\tau_1\tau_2 \equiv \det(\boldsymbol{\tau}) = \det(\mathbf{s}), \tag{28}$$

and so there are, correspondingly, two possible combinations of the signs of τ_1 and τ_2 (associated with two distinct branches of the inversion of (4)). Hence the ambiguity in the sign of $\bar{p}$. For a fuller discussion of the invertibility of the relation $\mathbf{s}(\boldsymbol{\alpha})$ the reader is referred to [5]. We remark in passing that it is convenient here to set $\tau_3 = s_{33}$.

From the definition of $\boldsymbol{\tau}$ it is seen that the scalar product $\boldsymbol{\tau}\mathbf{u}$ can also be written as $\mathbf{s}\boldsymbol{\alpha} \equiv s_{\mu i}\alpha_{i\mu}$ (with summation over μ and i from 1 to 2), and hence, by use of (12), (14), (15), (19) and (25), we find

$$\boldsymbol{\tau}\mathbf{u} \equiv \mathbf{s}\boldsymbol{\alpha} = pW_p + qW_q = \tfrac{1}{2}(p\bar{p}+q\bar{q}). \tag{29}$$

We now define the complementary energy function $\bar{W}$ through

$$W + \bar{W} = \boldsymbol{\tau}\mathbf{u}, \tag{30}$$

bearing in mind (29). $\bar{W}$ is regarded as a function of $\boldsymbol{\tau}$, depending on $\boldsymbol{\tau}$ only through $\bar{p}$ and $\bar{q}$ since the material is isotropic, but it may also be regarded as a function of s through (20).

The existence of $\bar{W}$ is assured where $\boldsymbol{\tau}(\mathbf{u})$ is locally invertible, that is where $\det(\partial\boldsymbol{\tau}/\partial\mathbf{u}) \neq 0$. This condition certainly holds within some domain of $\mathbf{u}$-space enclosing the origin (the undeformed configuration) since $\partial\boldsymbol{\tau}/\partial\mathbf{u}$ is positive definite there[5]. And, as discussed in [5], this domain extends at least to values of $\mathbf{u}$ associated with elastic failure, and positive definiteness of $\partial\boldsymbol{\tau}/\partial\mathbf{u}$ can therefore be regarded as a constitutive assumption in respect of the elastic response of solids. Furthermore, the domain can be taken to be sufficiently extensive so as to ensure global convexity of $\boldsymbol{\tau}(\mathbf{u})$ in the domain of elastic response. This means that $\boldsymbol{\tau}(\mathbf{u})$ is *globally* invertible within the elastic domain[5, 6].

In view of the above remarks we deduce that $\bar{W}$ exists, as does the inversion

$$\mathbf{u} = \frac{\partial \bar{W}}{\partial \boldsymbol{\tau}} \tag{31}$$

within the elastic domain. Moreover, the inversion (31) is unique. This does not, however, rule out the possibility of the existence of more than one solution to a given boundary-value problem. Nor does it rule out the possibility of buckling and certain types of instability. In practice, when considering the solution of actual boundary-value problems, some criterion of rupture or yield may be needed in order to determine whether or not the $\boldsymbol{\tau}$'s associated with a given $\mathbf{s}$ are within the elastic domain. Which $\boldsymbol{\tau}$ is the relevant one in a given problem will usually be clear from the context.

In terms of principal components (31) reduces to

$$\lambda_1 = \frac{\partial \bar{W}}{\partial \tau_1}, \lambda_2 = \frac{\partial \bar{W}}{\partial \tau_2} \tag{32}$$

or, in terms of invariants,

$$\tfrac{1}{2}p = \bar{W}_{\bar{p}}, \tfrac{1}{2}q = \bar{W}_{\bar{q}}, \tag{33}$$

from (25), (29) and (30), these being the inverse of (25).

Now, for isotropic elastic solids in plane strain $\partial\boldsymbol{\tau}/\partial\mathbf{u}$ is positive definite if and only if

$$\text{(a)}\ \ \partial\tau_i/\partial\lambda_j \text{ is positive definite } (i, j = 1 \text{ or } 2) \tag{34}$$

and

$$\text{(b)}\ \ (\tau_1 - \tau_2)(\lambda_1 - \lambda_2) > 0, \lambda_1 \neq \lambda_2, \tag{35}$$

as discussed more generally by Ogden[5].

From (17) expressed in terms of p and q it is easily shown that (35) becomes

$$W_q \geq 0, \tag{36}$$

with equality if and only if $\lambda_1 = \lambda_2$. Hence the definition $(21)_2$ of $\bar{q}$ as a positive quantity, in the light of $(25)_2$. On the other hand $W_p = \frac{1}{2}\bar{p} = \frac{1}{2}(\tau_1 + \tau_2)$ may be either positive or negative, since (34) puts a restriction on the second derivatives of W, namely $W_{pp}W_{qq} - W_{pq}^2 > 0$.

Since χ and χ^* are determined (to within an integral multiple of π) by (26) and (27) respectively when $\mathbf{s}$ is given, the inverse constitutive law can be written as

$$\boldsymbol{\alpha} = \frac{\partial \bar{W}}{\partial \mathbf{s}}, \tag{37}$$

(at least two distinct $\boldsymbol{\alpha}$'s corresponding to a given $\mathbf{s}$ in the global sense) *provided firstly that* $\partial\boldsymbol{\tau}/\partial\mathbf{u}$ *is non-singular* (as is certainly the case within the domain where it is positive definite) *and secondly that*

$$W_p \equiv \tfrac{1}{2}\bar{p} \equiv \tfrac{1}{2}(\tau_1 + \tau_2) \neq 0, \tag{38}$$

a requirement previously noted in (18).

With the help of (20) and (22), (37) can be written as

$$\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix} = \bar{W}_{\bar{p}}\begin{pmatrix}\cos\chi & \sin\chi\\ -\sin\chi & \cos\chi\end{pmatrix} + \bar{W}_{\bar{q}}\begin{pmatrix}\cos\chi^* & \sin\chi^*\\ \sin\chi^* & -\cos\chi^*\end{pmatrix}. \tag{39}$$

This may be compared with (12). *It is the decoupling of the relations between invariants, as expressed by* (25) *and* (33), *which is the particular feature here enabling the inverse to be written in the form* (39). We recall, however, that χ and χ^* are known only to within an integral multiple of π when s is given. When s is given the sign of $\bar{p}$ is ambiguous and this fact is reflected in the values of $\bar{W}_{\bar{p}}$ and $\bar{W}_{\bar{q}}$ since there are two distinct branches of $\bar{W}$ when it is viewed as a function of s.

From (33) and the definitions of p and q we have

$$\bar{W}_{\bar{p}} > 0,\ \bar{W}_{\bar{q}} \geq 0 \tag{40}$$

with equality if and only if $q = 0$.

From (13), (23), (25) and (33) we obtain

$$\left.\begin{aligned}\alpha_{11}+\alpha_{22} = 4A\bar{W}_{\bar{p}}/\bar{p},\ \alpha_{12}-\alpha_{21} = 4B\bar{W}_{\bar{p}}/\bar{p},\\ \alpha_{11}-\alpha_{22} = 4C\bar{W}_{\bar{q}}/\bar{q},\ \alpha_{12}+\alpha_{21} = 4D\bar{W}_{\bar{q}}/\bar{q},\end{aligned}\right\} \tag{41}$$

provided $\bar{p} \neq 0$.

The right-hand sides of eqns (41) depend only on A, B, C and D. When these are known and when $\bar{W}$ can be calculated, the components of $\boldsymbol{\alpha}$ are given by (41) and they must satisfy the compatibility relations (6).

By introducing the notation

$$P = \bar{W}_{\bar{p}}/\bar{p},\ Q = \bar{W}_{\bar{q}}/\bar{q} \tag{42}$$

we can rewrite (41) as

$$\left.\begin{aligned}\alpha_{11} = 2(PA+QC),\ \alpha_{12} = 2(PB+QD),\\ \alpha_{21} = 2(-PB+QD),\ \alpha_{22} = 2(PA-QC).\end{aligned}\right\} \tag{43}$$

The basic problem expressed by eqns (5) and (6) is now replaced by four first-order partial differential equations in A, B, C and D; thus

$$\left.\begin{aligned}(A+C)_{,1}+(B+D)_{,2} = 0,\\ (-B+D)_{,1}+(A-C)_{,2} = 0,\end{aligned}\right\} \tag{44}$$

$$\left.\begin{aligned}(PA+QC)_{,2}-(PB+QD)_{,1} = 0,\\ (-PB+QD)_{,2}-(PA-QC)_{,1} = 0.\end{aligned}\right\} \tag{45}$$

There are two possibilities for the sign of $\bar{p}$ for a given s and hence for the sign of P. We recall that P and Q are known functions of A, B, C and D from (25) and (42).

If the boundary of the plane region occupied by the material is the curve Σ in the undeformed configuration and the unit tangent to Σ is $(\cos\psi, \sin\psi)$, then the components (t_1, t_2) of traction per unit area are given by

$$\left.\begin{aligned}t_1 &= W_p\sin(\psi-\chi)+W_q\sin(\psi-\chi^*) = n_1(A+C)+n_2(B+D),\\ t_2 &= -W_p\cos(\psi-\chi)+W_q\cos(\psi-\chi^*) = n_1(-B+D)+n_2(A-C),\end{aligned}\right\} \tag{46}$$

n_1 and n_2 being the components of the unit normal to Σ.

The functions A, B, C and D must be consistent with the traction boundary conditions on Σ, and the solution of (43) for x_1 and x_2 should satisfy the boundary conditions of place on the relevant part of Σ.

The equilibrium eqns (5) can be satisfied identically if stress functions h_1 and h_2 are introduced such that

$$s_{11} = h_{2,2}, s_{21} = -h_{2,1}, s_{12} = -h_{1,2}, s_{22} = h_{1,1}.$$

Then, from (24) we have

$$\left.\begin{aligned} 2A &= h_{1,1} + h_{2,2}, 2B = h_{1,2} - h_{2,1}, \\ 2C &= h_{2,2} - h_{1,1}, 2D = -h_{1,2} - h_{2,1}, \end{aligned}\right\} \tag{47}$$

and we note that

$$\left.\begin{aligned} 2(A_{,1} + B_{,2}) &= -2(C_{,1} + D_{,2}) = \nabla^2 h_1, \\ 2(A_{,2} - B_{,1}) &= \quad 2(C_{,2} - D_{,1}) = \nabla^2 h_2, \end{aligned}\right\} \tag{48}$$

where $\nabla^2 = \partial^2/\partial X_1{}^2 + \partial^2/\partial X_2{}^2$.

For a given form of W the problem may now be reduced to finding two functions h_1 and h_2 satisfying the compatibility conditions (45) and consistent with the traction boundary conditions. Alternatively, assumptions can be made about h_1 and h_2, x_1 and x_2 then being found by integrating (43).

So far no assumptions have been made about the functions h_1, h_2, and hence A, B, C and D, and no restriction has been put on the form of W, although it is understood that its form is such that the definition of $\bar{W}$ is meaningful. It is possible, however, to make assumptions about h_1 and h_2 so that all the conditions of the considered problem are satisfied for particular classes of strain-energy functions W, thereby allowing explicit analytic solutions to be obtained. This is illustrated in Sections 4, 5 and 6 where h_1 and h_2 are assumed to be harmonic functions. This follows the parallel formulation of the dual problem in Section 3.

It is emphasised that the choice of h_1 and h_2 as harmonic functions is just one of many possible assumptions and is used here merely as an illustration of the general approach. Other choices, leading to different classes of W, are considered elsewhere, as also is the more direct approach in which no assumptions are made and the non-linear eqns (45) are solved for specific boundary-value problems. In fact we have obtained a more concise and simpler formulation of the basic equations in terms of complex variables. It has been used to obtain solutions for a number of boundary-value problems and the results will be reported in subsequent papers. In principle the results are obtainable for arbitrary forms of W, but in practice analytic solutions can be given in closed form only for relatively simple forms of W.

In the special case in which $\bar{p} = 0$ the angle χ is undetermined by s since the relation (26) is no longer valid. However, the inverse (39) can still be used provided χ is regarded as arbitrary. Since, in this case, $A = B = 0$ we must replace $2PA$ and $2PB$ respectively by $\bar{W}_{\bar{p}} \cos\chi$ and $\bar{W}_{\bar{p}} \sin\chi$ in (45). Equations (45), together with the boundary conditions then serve to determine χ.

From (47) we can then introduce the potential ϕ with $\nabla^2\phi = 0$ and such that

$$2C = \phi_{,22} - \phi_{,11}, D = -\phi_{,12}.$$

Then $\bar{W}_{\bar{p}}$ and $\bar{W}_{\bar{q}}$ depend only on $\bar{q} = (C^2 + D^2)^{1/2}$. The problem is now reduced to finding the harmonic function ϕ, together with χ from (45), consistent with the boundary conditions. The solution may not be unique.

3. THE DUAL FORMULATION

An alternative approach to that described in Section 2 is to begin by writing the components of the deformation gradient in the form

$$\left.\begin{aligned} \alpha_{11} &= \bar{A} + \bar{C}, \alpha_{21} = -\bar{B} + \bar{D}, \\ \alpha_{12} &= \bar{B} + \bar{D}, \alpha_{22} = \bar{A} - \bar{C}, \end{aligned}\right\} \tag{49}$$

where, bearing in mind (39),

$$\left.\begin{aligned} \bar{A} &= \bar{W}_{\bar{p}} \cos\chi, \quad \bar{B} = \bar{W}_{\bar{p}} \sin\chi, \\ \bar{C} &= \bar{W}_{\bar{q}} \cos\chi^*, \bar{D} = \bar{W}_{\bar{q}} \sin\chi^*. \end{aligned}\right\} \tag{50}$$

Also, we have

$$\tfrac{1}{2}p = \bar{W}_p = (\bar{A}^2 + \bar{B}^2)^{1/2}, \tfrac{1}{2}q = \bar{W}_q = (\bar{C}^2 + \bar{D}^2)^{1/2}. \tag{51}$$

From the compatibility equations, which are satisfied identically, we have, by analogy with (47),

$$\left.\begin{array}{l} 2\bar{A} = x_{1,1} + x_{2,2}, 2\bar{B} = x_{1,2} - x_{2,1}, \\ 2\bar{C} = x_{1,1} - x_{2,2}, 2\bar{D} = x_{1,2} + x_{2,1}, \end{array}\right\} \tag{52}$$

where the functions x_1 and x_2 are determined through the equilibrium equations (this is the conventional direct approach), and satisfy the boundary conditions of place on Σ.

The components of nominal stress are obtained from (12), with (50) and (51), in terms of $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$. Thus

$$\left.\begin{array}{l} s_{11} = 2(\bar{P}\bar{A} + \bar{Q}\bar{C}), s_{12} = 2(-\bar{P}\bar{B} + \bar{Q}\bar{D}), \\ s_{21} = 2(\bar{P}\bar{B} + \bar{Q}\bar{D}), s_{22} = 2(\bar{P}\bar{A} - \bar{Q}\bar{C}), \end{array}\right\} \tag{53}$$

where the notation

$$\bar{P} = W_p/p, \bar{Q} = W_q/q \tag{54}$$

has been introduced. In this notation the equilibrium equations are

$$\begin{array}{l} (\bar{P}\bar{A} + \bar{Q}\bar{C})_{,1} + (\bar{P}\bar{B} + \bar{Q}\bar{D})_{,2} = 0, \\ (-\bar{P}\bar{B} + \bar{Q}\bar{D})_{,1} + (\bar{P}\bar{A} - \bar{Q}\bar{C})_{,2} = 0. \end{array} \tag{55}$$

As an alternative to determining x_1 and x_2 directly from (55) with (52) (a formidable problem in general) it is possible to make assumptions about x_1 and x_2 (analogously to those for h_1 and h_2 in Section 2) and then (55) act as compatibility equations, possibly restricting the form of W which occurs through $\bar{P}$ and $\bar{Q}$.

The dual approach described here has the practical advantage that its validity is unrestricted and, moreover, it does not require a knowledge of the complementary-energy function $\bar{W}$, as is required in Section 2. Nevertheless it is the method of Section 2 which is used for illustration in this paper.

Finally, in this section, we note that

$$\left.\begin{array}{l} 2(\bar{A}_{,1} + \bar{B}_{,2}) = 2(\bar{C}_{,1} + \bar{D}_{,2}) \quad = \nabla^2 x_1, \\ 2(\bar{A}_{,2} - \bar{B}_{,1}) = -2(\bar{C}_{,2} - \bar{D}_{,1}) = \nabla^2 x_2, \end{array}\right\} \tag{56}$$

analogously to (48).

4. CONSEQUENCES OF THE ASSUMPTION THAT THE STRESS FUNCTIONS ARE HARMONIC

4.1 *Two special classes of strain-energy functions*

We now assume that h_1 and h_2 are harmonic, the motivation for this being eqns (48), so that

$$\begin{array}{l} A_{,1} + B_{,2} = 0, -B_{,1} + A_{,2} = 0, \\ C_{,1} + D_{,2} = 0, -D_{,1} + C_{,2} = 0. \end{array} \tag{57}$$

Thus (A, B) and (C, D) are conjugate harmonic pairs of functions, and we can write

$$A = \phi_{,1} = \psi_{,2}, B = \phi_{,2} = -\psi_{,1}, C = \phi^*_{,1} = \psi^*_{,2}, D = \phi^*_{,2} = -\psi^*_{,1}, \tag{58}$$

where

$$\nabla^2\phi = \nabla^2\phi^* = \nabla^2\psi = \nabla^2\psi^* = 0. \tag{59}$$

Moreover, we note that a consequence of (57) is that χ and χ^* are harmonic, as may be demonstrated by using (26) and (27).

With the help of (57) it can be shown that the compatibility eqns (45) can be written as

$$\left.\begin{aligned}\nabla P\cdot\nabla\phi-\nabla Q\cdot\nabla\phi^*=0,\\ \nabla P\wedge\nabla\phi+\nabla Q\wedge\nabla\phi^*=0,\end{aligned}\right\}\tag{60}$$

(or similarly in terms of ψ and ψ^*) in the usual notation for scalar and vector products of two vectors, ∇ representing the (plane) vector operator with components $\partial/\partial X_1$, $\partial/\partial X_2$.

The components of traction t_1 and t_2 on Σ can be written

$$t_1=\mathbf{m}\cdot\nabla(\psi+\psi^*),\ t_2=\mathbf{m}\cdot\nabla(\phi^*-\phi),$$

where $\mathbf{m}$ is the unit tangent to Σ measured in the positive sense. Specification of the traction on Σ can therefore be regarded as equivalent to specifying $\psi+\psi^*$ and $\phi^*-\phi$ which, incidentally, are just the stress functions h_2 and $-h_1$ respectively.

The quantities P and Q appearing in (60) and given by (25) are both functions of $|\nabla\phi|$ and $|\nabla\phi^*|$ in view of (58). *In general, the conditions* (60) *are incompatible with ϕ and ϕ^* being harmonic.* This, of course, reflects the fact that we are making *assumptions* about h_1 and h_2, indicating that there do not exist (in general) solutions of the assumed type. In certain circumstances, however, eqns (60) are entirely consistent with ϕ and ϕ^* being harmonic. A trivial case in which this is so is when p and q (and hence $\bar{p}$, $\bar{q}$, χ and χ^*) are constant, and the deformation is homogeneous. It may be recalled that any homogeneous deformation can be maintained in an elastic solid by the application of suitable surface tractions.

We now consider two particular classes of strain-energy function for which (60) can be satisfied in non-trivial situations.

Class I. This class of materials has strain-energy function given by

$$W=f(p)+\tfrac{1}{2}\mu q^2,\tag{61}$$

when μ is a constant and f is some function whose behaviour need not be specified at this point except that it should be twice continuously differentiable. When rearranged in the form

$$W=F(p)-\tfrac{1}{2}\mu(p^2-q^2)\tag{62}$$

this is recognizable as the class of *harmonic materials* introduced by John[4]. John's approach to the problem, however, is somewhat different than that considered here, and comparisons between these approaches for this particular class of materials are discussed in Section 5.

From (61), with the help of (25), (33) and (42), we find

$$Q=\frac{\bar{W}_{\bar{q}}}{\bar{q}}=\frac{1}{4\mu},$$

and hence $\nabla Q=0$. The consequence of (60) is then that

$$\nabla P=0,\tag{63}$$

provided $\nabla\phi\neq 0$. It follows that *either p is constant, or W has the form*

$$W=\text{constant}\times p^2+\tfrac{1}{2}\mu q^2.$$

However, this particular form of W is not consistent with the usual linear theory for infinitesimal deformations unless the material is prestressed by an all-round hydrostatic stress. To comply with the linear theory the function $f(p)$ in (61) must be such that

$$f(2)=0,f'(2)=0,f''(2)=\lambda+\mu,\tag{64}$$

where λ and μ are the Lamé constants and where $f'(p) = df/dp$. The *semi-linear material*† with strain energy

$$W = \tfrac{1}{2}(\lambda + \mu)(p - 2)^2 + \tfrac{1}{2}\mu q^2 \tag{65}$$

satisfies the condition (64).

For any member of the class (61) consistent with (64) we have (63), which gives

$$p = \text{constant}, \tag{66}$$

thus restricting the type of solution possible under our assumptions. This does not mean, of course, that p must always be constant for materials of type (61). It merely indicates that our assumption of h_1 and h_2 being harmonic functions is only appropriate for those deformations for which p happens to be constant. Other choices of h_1 and h_2 will lead to different restrictions, if any, on p and q and therefore provide solutions to problems involving different types of deformation.

In passing we remark that the inequalities $\mu > 0$ and $f''(p) > 0$ are necessary and sufficient for (34) and (35) to hold in respect of this material.

Class II. We now consider the class of materials with strain-energy functions of the form

$$W = \tfrac{1}{2}\eta p^2 + g(q), \tag{67}$$

where η is a constant and $g(q)$ satisfies the conditions

$$g(0) = g'(0) = 0, g''(0) = \mu - \eta,$$

where μ is the classical shear modulus.

For these materials $\nabla P = 0$, the compatibility equations yield $\nabla Q = 0$ for a non-trivial solution and, provided $g(q)$ is not of the form $\tfrac{1}{2}(\mu - \eta)q^2$, we deduce that

$$q = \text{constant}. \tag{68}$$

This class of materials, however, is of only marginal interest since the material is subject to a hydrostatic pre-stress of amount 2η, and, furthermore, when reduced to the linear theory it is found that the Lamé constants are such that $\lambda + \mu = 0$.

For future reference we write (67) in the form

$$W = \tfrac{1}{2}\eta(p^2 - q^2) + G(q). \tag{69}$$

We note that the inequalities $\eta > 0$, $g''(q) > 0$ and $g'(q) \geq 0$ (with equality if and only if $q = 0$) are necessary and sufficient for (34) and (35) to hold for class II materials.

We have seen, therefore, that for two classes of materials we can obtain solutions to problems in which invariant p or q is constant. If different initial assumptions are made about h_1 and h_2 we can expect to obtain solutions for other classes of strain-energy functions than those considered above. For example, a relationship between $\bar{p}$ and $\bar{q}$ may be imposed. On the other hand different solutions for the materials in classes I and II may be obtained by making different assumptions about h_1 and h_2 which do not impose restrictions on the invariants. For example, by rearranging the terms occurring in W in (61) or (67) it is possible to obtain such solutions, and, in principle, one can obtain solutions to any boundary-value problem for these materials. This is the basis of John's approach[4] when W has the form which is essentially (62). This is discussed further in Section 5.

Ideally, one would prefer not to have to make assumptions about h_1 and h_2, but keep them as general as possible for any given constitutive law. If the class I materials are considered, for example, it can be shown that the compatibility eqns (45) are expressible in the form

†This special material has been discussed in detail by, for example, Sensenig[7, 8].

$$\nabla^2\{(P+Q)(h_{1,2}-h_{2,1})\}=0, \nabla^2\{(P+Q)(h_{1,1}+h_{2,2})\}=0,$$

with the help of (47) and (48). Such a general approach may be amenable to analysis for certain types of problem, but will be pursued elsewhere.

4.2 *Circular symmetry*

We can retain W in its most general form as a function of p and q and consider the resulting equations for the case of a circular cylindrical deformation. This throws some light on what forms of W can be admitted under the assumptions of harmonic h_1 and h_2.

In the case of circular symmetry the deformation gradient is symmetric and, from (26), we deduce that $\chi=0$. It follows from (23) that $B=0$ and hence, from (57), that A is constant.

Thus, from (25) and (58),

$$\left.\begin{aligned}\tfrac{1}{2}\bar{p} &= W_p(p,q)=A=\text{constant},\\ \tfrac{1}{2}\bar{q} &= W_q(p,q)=(C^2+D^2)^{1/2}=|\nabla\phi^*|,\end{aligned}\right\} \tag{70}$$

where ϕ^* is harmonic.

Also, from (15), $\chi^*=2\theta$, where θ is the polar angle measured from the X_1-axis, and therefore

$$D=C\tan 2\theta$$

from (27). Equation $(70)_2$ then gives

$$\tfrac{1}{2}\bar{q}=C\sec 2\theta=\frac{CR^2}{X_1^2-X_2^2},$$

where $R=(X_1^2+X_2^2)^{1/2}$.

Because of the symmetry $\bar{q}$ must be a function of R only. We therefore deduce, since ϕ^* is harmonic, that

$$\phi^*=-\frac{\alpha X_1}{R^2}-\frac{\beta X_2}{R^2},$$

where α and β are constants. However, since the traction must be radial, it can be shown that we must have $\beta=0$, so that

$$(C^2+D^2)^{1/2}=\frac{\alpha}{R^2}$$

and the components of nominal stress are given by

$$s_{11}=A-\frac{\alpha(X_2^2-X_1^2)}{R^4}, s_{12}=s_{21}=\frac{2\alpha X_1X_2}{R^4}, s_{22}=A+\frac{\alpha(X_2^2-X_1^2)}{R^4},$$

independently of the form of W.

Since

$$P\equiv\frac{\bar{W}_{\bar{p}}}{\bar{p}}=\frac{p}{4A}$$

and

$$Q\equiv\frac{\bar{W}_{\bar{q}}}{\bar{q}}=\frac{qR^2}{4\alpha}$$

the compatibility eqns (45) (after some analysis, or more directly, starting from (39)) become

$$p'-q'-\frac{2q}{R}=0, \tag{71}$$

where $p'=\mathrm{d}p/\mathrm{d}R$.

Only those forms of W compatible with (71) and with

$$W_p(p, q) = A, \; W_p(p, q) = \alpha / R^2 \tag{72}$$

are admissible *under the assumption of harmonic h_1 and h_2.*

No attempt will be made here to list those forms of W for which (71) and (72) hold.

For materials in class I

$$q = \frac{\alpha}{\mu R^2}, \; p = (f')^{-1}(A)$$

where $(f')^{-1}$ is the inverse function of f, which is assumed to exist. (This assumption is justified in Section 7). The deformation can be written

$$x_i = rX_i / R \; (i = 1, 2),$$

where

$$r = \{1 + \tfrac{1}{2}(f')^{-1}(A)\}R - \frac{\alpha}{2\mu R} \tag{73}$$

We deduce from (73), in particular, that for a solid circular cylinder the deformation is necessarily homogeneous ($\alpha = 0$), a result analogous to that known in the linear theory.

For class II materials (72) gives

$$p = \frac{A}{\eta}, \; g'(q) = \frac{\alpha}{R^2}$$

but these are incompatible with (71) except in the trivial case $g'(q) = (\mu - \eta)q$. Therefore, the assumptions of harmonic h_1 and h_2 are not appropriate for circular symmetric deformations in class II materials.

It is not intended that too much emphasis should be put on the specific choice of harmonic functions in this paper. The choice is made here merely to illustrate the possibilities afforded by the more general approach in Sections 2 or 3.

5. COMPARISON WITH JOHN'S FORMULATION

John [4] considered plane strain problems for materials with a strain-energy function which, in the present notation, may be written as (61) or as

$$W = F(p) - 2\mu J, \tag{74}$$

where J is given by (1).

Use of (3), (10) and (13) enables us to write the components of s as

$$s_{11} = F'(p)\cos\chi - 2\mu\alpha_{22}, \quad s_{21} = F'(p)\sin\chi + 2\mu\alpha_{21},$$
$$s_{12} = -F'(p)\sin\chi + 2\mu\alpha_{12}, \; s_{22} = F'(p)\cos\chi - 2\mu\alpha_{11}.$$

If we write

$$A = F'(p)\cos\chi, \; B = F'(p)\sin\chi, \tag{75}$$

(these being different from the A and B used earlier) the equilibrium equations reduce to

$$A_{,1} + B_{,2} = 0, \; A_{,2} - B_{,1} = 0 \tag{76}$$

on the assumption that the components of $\boldsymbol{\alpha}$ satisfy the compatibility eqns (6). Then, *the functions A and B are automatically conjugate harmonics.*

The components of x_1 and x_2 are then found from the eqns (75) expressed, with the help of (13), in the form

$$\begin{aligned} x_{1,1}+x_{2,2} &= pA/(A^2+B^2)^{1/2}, \\ x_{1,2}-x_{2,1} &= pB/(A^2+B^2)^{1/2}, \end{aligned} \tag{77}$$

where

$$p=(F')^{-1}\{(A^2+B^2)^{1/2}\}.$$

It is therefore $F'(p)\cos\chi$ and $F'(p)\sin\chi$ which are conjugate harmonics in John's method, whereas $\{F'(p)-\mu p\}\cos\chi$ and $\{F'(p)-\mu p\}\sin\chi$ are the (assumed) conjugate harmonics in the approach of Section 4.

We note that the components x_1 and x_2 can be expressed, quite generally, in the form

$$x_1=\phi_{,1}+\psi_{,2},\; x_2=\phi_{,2}-\psi_{,1}, \tag{78}$$

where ϕ and ψ are scalar functions. The solution of eqns (77) is then reduced to the solution of two Poisson equations, namely

$$\nabla^2\phi=pA/(A^2+B^2)^{1/2},\nabla^2\psi=pB/(A^2+B^2)^{1/2}. \tag{79}$$

It should be pointed out here that John's method, as it should, gives the same solution to the circular symmetric problem as the method described in Section 4.2. It turns out that p is constant. That p is constant is not immediately obvious from John's method, and in this respect the method of Section 4 has an advantage over that of John for the class I materials. However, *for these materials* John's method is more general and deserves to have received more attention than it has up to the present. On the other hand the approach of Sections 2 and 3 offers scope for the solution of problems in plane-strain finite elasticity for wider classes of strain-energy function.

6. THE CIRCULAR SHEAR PROBLEM

From (46) the components of traction on the boundary Σ can be expressed in the complex form

$$t_1-it_2=(A+iB)(n_1-in_2)+(C-iD)(n_1+in_2) \tag{80}$$

per unit area in the undeformed configuration.

The components x_1 and x_2 are obtained by integrating the eqns (43) which, for the class I materials (61) to which attention is now restricted, are expressible as

$$\begin{aligned} \alpha_{11}&=2PA+C/2\mu, \quad \alpha_{12}=2PB+D/2\mu, \\ \alpha_{21}&=-2PB+D/2\mu, \alpha_{22}=2PA-C/2\mu. \end{aligned} \tag{81}$$

From (25) we have

$$\tfrac{1}{2}\bar{p}=f'(p)=(A^2+B^2)^{1/2}, \tfrac{1}{2}\bar{q}=\mu q=(C^2+D^2)^{1/2},$$

and, according to the method of Section 4, p is constant. Hence, by means of (57) it follows that χ, A and B are constant. With the help of (58), the integral of (81) can be expressed in the form

$$\bar{z}=2P(A+iB)\bar{\zeta}+\tfrac{1}{2}\omega(\zeta)/\mu+\text{constant}, \tag{82}$$

where

$$\omega(\zeta)=\phi^*+i\psi^*$$

is an analytic function of $\zeta = X_1 + iX_2 = Re^{i\Theta}$, and

$$C - iD = \frac{d\omega}{d\zeta}. \tag{83}$$

Also $\bar{\zeta} = X_1 - iX_2$ and $z = x_1 + ix_2 = re^{i\theta}$

Since the components x_1 and x_2 are single valued $\omega(\zeta)$ can be written

$$\omega(\zeta) = \sum_{-\infty}^{\infty} a_n \zeta^n, \tag{84}$$

where the a_n's are constants.

We now consider an elastic solid contained in the annular region $a \le R \le b$ in the underformed configuration, so that on any circle concentric with the annulus $n_1 + in_2 = e^{i\Theta}$.

In polar coordinates (R, Θ) in the undeformed configuration the components of tractions t_R and t_Θ on a circle of radius R are given by

$$t_R - it_\Theta = (t_1 - it_2)(n_1 + in_2) = A + iB + (C - iD)(n_1 + in_2)^2. \tag{85}$$

We now suppose that the inner surface $R = a$ is held fixed while traction $t_\Theta = S$ is applied to $R = b$. The boundary conditions are therefore

$$\left.\begin{aligned} \bar{z} = \bar{\zeta} \quad &\text{on} \quad R = a, \\ t_R - it_\Theta = -iS \quad &\text{on} \quad R = b. \end{aligned}\right\} \tag{86}$$

Substitution of (82), (83) and (84) into (86) yields

$$\left.\begin{aligned} \omega = a_0 + a_{-1}\zeta^{-1}, C - iD = -a_{-1}\zeta^{-2}, \\ \bar{z} = 2P(A + iB)\bar{\zeta} + a_{-1}\zeta^{-1}/2\mu, \end{aligned}\right\} \tag{87}$$

the constant term in $\bar{z}$ having been set to zero since it corresponds to a rigid displacement. Also we have

$$a_{-1} = 2\mu a^2\{1 - 2P(A + iB)\} = b^2\{A + i(B + S)\}. \tag{88}$$

Equations (88) are enough to determine A and B when S is given since

$$P = (f')^{-1}\{(A^2 + B^2)^{1/2}\}/4(A^2 + B^2)^{1/2} \tag{89}$$

The deformation can be described in the form

$$r^2 = \left\{\frac{a^2}{R} + 2PA\left(R - \frac{a^2}{R}\right)\right\}^2 + \left\{R - \frac{a^2}{R}\right\}^2 4P^2B^2, \tag{90}$$

$$\tan(\theta - \Theta) = -2PB\left(R - \frac{a^2}{R}\right) \Big/ \left\{\frac{a^2}{R} + 2PA\left(R - \frac{a^2}{R}\right)\right\}. \tag{91}$$

The form of solution can be simplified if one applies, in addition to the shear stress S, a radial stress t_R on $R = b$ which happens to equal the value of A (which is initially undetermined). Then

$$a_{-1} = i\alpha, \alpha = (S + B)b^2 = -4\mu PBa^2, 2PA = 1,$$

and (90) and (91) are replaced by

$$r^2 = R^2 + \frac{\alpha^2}{4\mu^2}\left(\frac{1}{R} - \frac{R}{a^2}\right)^2, \tag{92}$$

$$\tan(\theta - \Theta) = \frac{\alpha}{2\mu}\left(\frac{1}{a^2} - \frac{1}{R^2}\right). \tag{93}$$

It is worth noting from (92) in particular that $r \geq R$ with equality holding only on $R = a$. Thus, the effect of the circular shear combined with a radial stress $t_R = A$ is to increase the volume of the material. Whether the volume increases or decreases when $t_R = 0$ on $R = b$ depends on the particular form of $f(p)$ and may be assessed by determining whether $r(b) > b$ or $< b$ from (90) with (88) and (89).

Calculations for particular forms of $f(p)$ are presented in Section 7.

For comparison we note that the corresponding results for the linear theory are expressible in the form

$$u_r = 0,\ u_\theta = \frac{b^2 S}{2\mu}\left(\frac{r}{a^2} - \frac{1}{r}\right),$$

where u_r and u_θ are the displacement components in polar coordinates, and the components of Cauchy stress are

$$\sigma_{rr} = \sigma_{\theta\theta} = 0,\ \sigma_{r\theta} = \frac{b^2 S}{r^2}.$$

We remark that the deformation is necessarily isochoric under these boundary conditions.

It is worth noting that, under the assumptions of harmonic h_1 and h_2 the condition $r = R$ cannot be imposed since this would imply a trivial solution $a_{-1} = 0$, $A = B = C = D = 0$. Thus, a volume preserving solution is not obtainable under the assumptions of the present method, that is an isochoric deformation is not compatible with p = constant other than in the trivial situation.

If John's method is used the same results are obtained and eqns (77) may be integrated immediately if the assumption of constant p is made. Without this assumption, however, the analysis is not so straightforward.

7. INEQUALITIES, NUMERICAL RESULTS AND DISCUSSION

In order to illustrate the results of Section 6, in particular eqns (90) and (91), we choose the following form of $f(p)$:

$$f(p) = \tfrac{1}{2}(\lambda + \mu)(p - 2)^2 + \frac{1}{24}\nu(p - 2)^4, \tag{94}$$

so that

$$f'(p) = (\lambda + \mu)(p - 2) + \frac{1}{6}\nu(p - 2)^3, \tag{95}$$

where ν is a constant. This form is consistent with (64) and, moreover, satisfies the condition

$$f''(p) > 0$$

if $\lambda + \mu > 0$ and $\nu > 0$, which we assume to be the case. When these inequalities hold it follows that

$$f'(p) \gtreqless 0 \text{ according as } p \gtreqless 2,$$

ensuring that (plane) hydrostatic stress is positive (negative) in tension (compression). Invertibility of $f'(p)$ is also a consequence of these inequalities.

It is not proposed to discuss inequalities in detail here, although more could be said about restrictions required on f and its derivatives to ensure physically reasonable response. In some respects, therefore, (94) may be unrealistic. However, it serves to illustrate the theory for the moderate range of strains to be considered in this Section.

For the purposes of carrying out numerical calculations the following non-dimensional quantities are defined:

$$\lambda^* = \lambda/\mu,\ \nu^* = \nu/\mu,\ f^* = f/\mu,\ A^* = A/\mu,\ B^* = B/\mu,$$
$$\gamma^* = (A^{*2} + B^{*2})^{1/2},\ P^* = \mu P,\ S^* = S/\mu,\ R^* = R/a,$$
$$\eta = b^2/a^2.$$

The range of values of R^* is from 1 to $\eta^{1/2}$.

After some algebra (88) and (89) can now be written as

$$A^* = 2\eta^{-1}(1+4P^*\eta^{-1})^{-1},\ B^* = -S^*(1+4P^*\eta^{-1})^{-1}, \tag{96}$$

and

$$P^* = \frac{1}{4}\gamma^{*-1}\{(\eta^2 S^{*2}+4)^{1/2} - \eta\gamma^*\}. \tag{97}$$

The quantity γ^* is then determined from the equation

$$\gamma^* = f^{*\prime}(p), \tag{98}$$

where $p = (\eta^2 S^{*2}+4)^{1/2} - \eta\gamma^*$, for given η and S^*.

A^*, B^* and P^* are then calculated from (96) and (97).

In non-dimensional notation (90) and (91) become

$$r^2R^{-2} = \{R^{*-2} + 2P^*A^*(1-R^{*-2})\}^2 + 4(1-R^{*-2})^2P^{*2}B^{*2} \tag{99}$$

and

$$\tan(\theta - \Theta) = -2P^*B^*(R^* - R^{*-1})\{R^{*-1} + 2P^*A^*(R^* - R^{*-1})\}^{-1}. \tag{100}$$

Calculations have been carried out for $b/a = 2$ and 4 ($\eta = 4$ and 16) and for the range of S^* from 0 to 1.5 in steps of 0.1. Values $\lambda^* = 0.1$ and $\nu^* = 24$ of the material constants have been chosen to illustrate the character of the results. Within the limitations of the form (94) other choices of λ^* and ν^* are equally acceptable, the main features of the results not depending on their values.

Figure 1 shows r^2R^{-2} as a function of R^* for the case $\eta = 16$ and for $S^* = 0.1, 0.3, 0.5$ and 1.0. Physically r^2R^{-2} is a measure (per unit length perpendicular to the (1, 2)-plane) of the change in volume of the material enclosed within the circle $R =$ constant. Notice, in particular, that near $R = a$ ($R^* = 1$), where the boundary is fixed, there is a marked volume decrease.

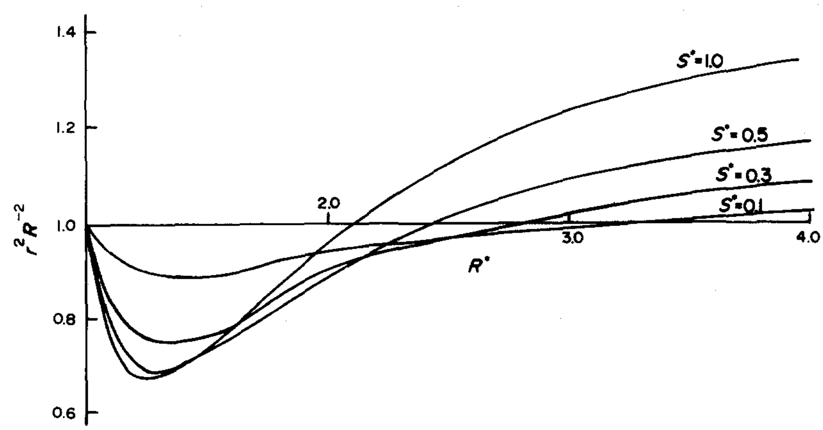

Fig. 1. Plot of r^2R^{-2} against R^* for $\eta = 16$.

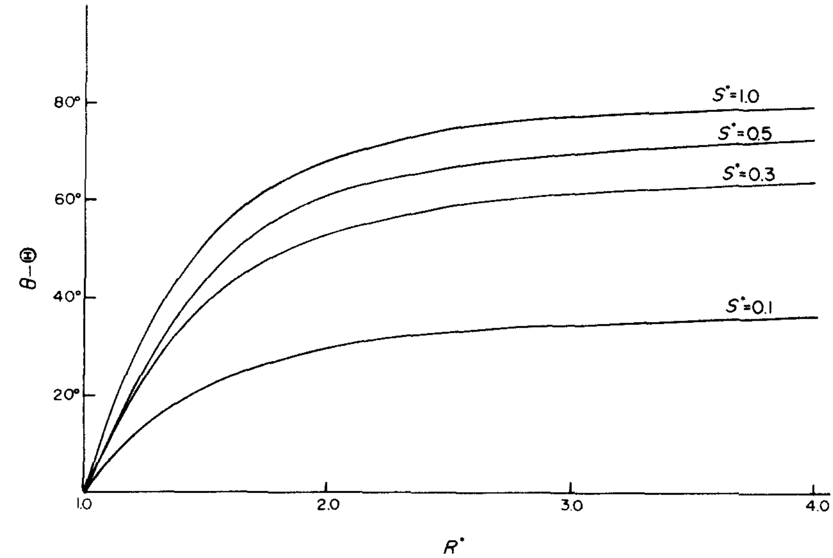

Fig. 2. Plot of $\theta - \Theta$ against R^* for $\eta = 16$.

This result is in fact independent of the form of $f(p)$ since

$$\frac{\mathrm{d}}{\mathrm{d}R^*}(r^2R^{-2}) = -2A^*\eta < 0$$

at $R^* = 1$. Moreover, r^2R^{-2} has a minimum value where

$$R^{*2} = \frac{4P^{*2}S^{*2}+1}{4P^{*2}S^{*2}-4P^*\eta^{-1}}.$$

The above behaviour is, of course, characteristic of r^2R^{-2} being quadratic in R^{*-2}, and the above value of R^* lies in the range $1 < R^* < \eta^{1/2}$ if $4P^{*2}S^{*2}(\eta-1) > 4P^*+1$.

One can also test whether the volume as a whole increases or decreases by determining if $r^2R^{-2} = 1$ has a solution for R^* in the range 1 to $\eta^{1/2}$. If it has the volume increases; if not, the volume decreases. The result depends on the form of $f(p)$, on η and on S^*. In the former case there is, for any S^*, a material circle whose radius equals its radius in the undeformed configuration.

In Fig. 1 the values chosen indicate an overall increase in volume, but an overall decrease in volume is not ruled out for other values. Indeed, for the same values of λ^* and ν^*, but for $\eta = 4$, our calculations show that there is an overall volume decrease for values of S^* less than about 0.5.

The actual magnitudes of the volume changes are up to 30%. This reflects the fact that no attempt has been made to find values of λ^* and ν^*, or, indeed, forms of $f(p)$, which are appropriate for any specific type of material. However, the shape of the curves is expected to be of the general character shown in Fig. 1 for materials such as rubber-like solids for which volume changes are of the order of 10^{-3} of those shown.

In Fig. 2 curves of $\theta - \Theta$ as a function of R^* are plotted for $S^* = 0.1$, 0.3, 0.5 and 1.0. They illustrate how the shearing is largest at $R = a$, decreasing monotonically with increasing R. This behaviour is to be expected in view of the fact that $R = a$ is a fixed boundary. The association of large shearing with decrease of radius near $R = a$ could also be anticipated.

From (85), (87) and (88) we obtain

$$t_R^* \equiv \mu^{-1}t_R = A^*(1-\eta R^{*-2}) \leq 0$$

and

$$t_\Theta^* \equiv \mu^{-1}t_\Theta = -B^* - 4P^*B^*R^{*-2} > 0$$

when $S^*>0$. Note, in particular, that t^*_R is compressive, and near $R^*=1$ is large, being associated with the decrease in radius of material circles in that neighbourhood. Nearer to $R^*=4$ the effect of the compressive radial stress is subordinate to the effect of the shear stress, resulting in volume expansion.

It is interesting to obtain an estimate of the magnitudes of the stresses at the inner boundary of the annulus. Using the results for $\eta=16$ we consider just one value of S^*, namely 0.1. At $R^*=1$ we obtain $t^*_R \simeq -0.37$, $t^*_\Theta \simeq 1.2$. It is noteworthy that the shear stress has increased to twelve times its value at $R^*=4$.

The values of the associated principal stretches may also be estimated. If we take $\lambda_1 \geq \lambda_2$, then

$$\lambda_1 = \tfrac{1}{2}(p+q),\ \lambda_2 = \tfrac{1}{2}(p-q),$$

where p and q are given by

$$p = (f^{*\prime})^{-1}(\gamma^*),\ q = (C^{*2}+D^{*2})^{1/2} = 2\{(1-2P^*A^*)^2 + 4P^{*2}B^{*2}\}^{1/2}R^{*-2}.$$

For $S^*=0.1$ we find that λ_1 has its maximum of approximately 1.7 at $R^*=1$ decreasing monotonically to $\simeq 1.06$ at $R^*=4$, whereas λ_2 increases monotonically from $\simeq 0.36$ at $R^*=1$ to $\simeq 0.98$ at $R^*=4$.

For higher values of S^* our calculations show that λ_2 can be negative and zero near $R^*=1$, which is physically unrealistic. This is a consequence of the particular form we have chosen for $f(p)$. To avoid this type of inconsistency further restrictions must be put on the form of $f(p)$.

Indeed, the inequality

$$f'(p) < \mu p \qquad (101)$$

must be satisfied in order that there is a positive value of λ_2 associated with an arbitrary $\lambda_1 > 1$ in pure shear. In different notation this condition was noted by John[4] for pure shear.

For the present $f(p)$ the range of validity of (101) can be extended to higher values of λ_1 by reducing the value of ν^*. Moreover, for the circular shear problem, positive values of λ_2 will emerge for higher values of S^* when ν^* is reduced.

Full discussion of (101) and other inequalities, along with consideration of other particular forms of $f(p)$ in relation to these inequalities and to data on actual material behaviour, is deferred.

Finally, we have one reservation concerning the form of constitutive law given by (61) in relation to eqn (91). As S^* increases from zero to infinity $\tan(\theta-\Theta)$ (evaluated for $R=b$) does likewise; it cannot be negative. Therefore, $\theta-\Theta$ increases from zero asymptotically to $\frac{1}{2}\pi$. Thus, the constitutive law (61) puts an upper limit on the angle through which material points can turn. This is unrealistic and in this sense a limitation is put on the range of deformations for which (61) can be used. For practical purposes, however, the range of validity is likely to be sufficient.

Acknowledgements—D. A. Isherwood is grateful for the support of an S.R.C. Research Studentship. The authors wish to thank R. J. Bedding for carrying out some numerical calculations.

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